Almost Monochromatic Triangles And Other Ramsey Problem Variants

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Abstract

In this paper, variants of the Ramsey Number problem are investigated. In particular, chromatic graphs not containing almost monochromatic triangles for color spectrum and color wheel, finite metric spaces not containing almost equilateral triangles and the relations between these different problems and the original Ramsey problem are studied.

1 Introduction

By way of generalizing the original Ramsey problem, we can use the intuitive concept of nearness of colors in a color wheel or a color spectrum. An explicit definition of this concept may be given as follows:

Definition 1. *Neighboring Colors*

Given a set of n colors in specific order, $\{k_1, k_2, \ldots, k_i, \ldots, k_j, \ldots, k_n\}$, two colors k_i and k_j, $(i < j)$ are said to be neighboring colors if $j - i = 1$. Additionally,

1. *In a color spectrum, k_n and k_1 are not defined to be neighboring.*
2. *In a color wheel, k_n and k_1 are defined to be neighboring.*

Definition 2. *Almost Monochromatic Triangles*

An almost monochromatic triangle is defined as a triangle in which all the colors of the sides are the same or neighbors of each other.

The Ramsey problem can thus be generalized: Determine the minimum number of points, so that when all points are connected to each other and the edges colored using n colors, the graph contains at least one almost monochromatic triangle, in the two cases of a color spectrum and a color wheel.

Definition 3. A_n

*A_n denotes the minimum number of points, so that when all points are connected to each other and the edges colored using n colors forming a **color spectrum**, the graph contains at least one almost monochromatic triangle.*

Definition 4. B_n

*B_n denotes the minimum number of points, so that when all points are connected to each other and the edges colored using n colors forming a **color wheel**, the graph contains at least one almost monochromatic triangle.*

In the following sections, further properties of A_n and B_n and ways of determining them as defined above for different values of n will be proposed.

2 Almost Monochromatic Triangles in a Color Spectrum

Theorem 5.

$$A_n \leq 2A_{n-2} + 2A_{n-3} + \sum_{i=3}^{n-2} R_2[A_{i-2},\ A_{n-i-1};\ 2] - n + 2,$$
$$A_1 = 3,\ A_2 = 3,\ A_3 = 6.$$

Proof. The result can be proved by anticipating the Pigeonhole Principle. Consider a list of n colors k_1, k_2,...,k_n. Fix a single point P of the graph. The maximum number of points that can be connected to P using color k_1 is given by $A_{n-2}-1$ each, as $n-2$ colors k_3, k_4,..., k_n remain which are not neighbors of k_1. If two points joined to P by k_1 were joined to each other by a neighboring color, we would get an almost monochromatic triangle. It is also the number of points for k_n. Similarly, for the second and second to last colors, it is $A_{n-3}-1$ each. If we select the i^{th} color k_i $(3 \le i \le n-2)$, we get two sets of $i-2$ (k_1, k_2,..., k_{i-2}) and $n-i-1$ (k_{i+2}, k_{i+3},..., k_n) colors which are not neighbors of k_i. The points joined to P cannot form a $A_{i-2}-$clique in $i-2$ colors, nor can they form a $A_{n-i-1}-$clique in $n-i-1$ colors. For, indeed, if there did, we could have a subgraph of A_{i-2} points in $i-2$ colors, or a subgraph of A_{n-i-1} points in $n-i-1$ colors, arranged in a color spectrum, which would guarantee an almost monochromatic triangle. So the maximum number of points joined to P by color k_i is $R_2[A_{i-2},\ A_{n-i-1};\ 2]-1$. This is true for all colors k_i such that $3 \le i \le n-2$. Therefore, the maximum number of edges which can join P to other points is

$$2(A_{n-2}-1)+2(A_{n-3}-1) + \sum_{i=3}^{n-2}(R_2[A_{i-2},\ A_{n-i-1};\ 2]-1).$$

Adding one more point would guarantee an almost monochromatic triangle. Therefore, including P in the count we get our desired result.

It is obvious that $A_1 = 3$, $A_2 = 3$. For A_3, consider a graph of 5 points joined to P. If we have two edges each of k_1 and k_3 and one edge of k_2 connected to P. As similar logic applies to k_1 and k_3, without loss of generality, consider the points connected to P by k_1 as A and B, and that connected by k_2 as C. Now A and B can only be connected to each other by k_3. This is also the case with A and C as well as B and C. Thus A, B and C form a monochromatic triangle. So, a graph of 6 points in three colors always gives an almost monochromatic triangle. Also, it is possible to construct a 5-point graph containing only the first and third colors, which are not neighbors, as this is equivalent to the case of the Ramsey Number $R_n[3,\ 3;\ 2]$. So $A_3 = 6$. ☐

The values of upper bounds calculated from the result of Theorem 5 (for $n > 8$, the upper bounds of the necessary Ramsey Numbers are as yet unknown):

$$A_4 \le 10$$
$$A_5 \le 21$$
$$A_6 \le 40$$
$$A_7 \le 99$$
$$A_8 \le 238.$$

3 Almost Monochromatic Triangles in a Color Wheel

The following relation between B_n and A_n can be proved:

Theorem 6.

$$B_n \le n(A_{n-3}-1)+2,\ B_1 = 3,\ B_2 = 3,\ B_3 = 5. \tag{1}$$

Proof. Consider a graph of $N = n(A_{n-3}-1)+2$ points. Suppose it does not contain an almost monochromatic triangle. Consider all edges connecting one point P to others. There are $N-1$ edges, in n colors. Applying the Pigeonhole Principle, we get that at least one color, say k_1, connects P to A_{n-3} points. Now, the edges interconnecting these A_{n-3} points cannot be neighbors of k_n. That leaves $n-3$ colors, excluding k_n, k_1 and k_2. But these $n-3$ colors form a color spectrum, and A_{n-3} points in $n-3$ colors forming a color spectrum guarantee an almost monochromatic triangle. This is a contradiction.

Thus, $N = n(A_{n-3}-1)+2$ points in n colors of a color wheel contains an almost monochromatic triangle.

Again, it is obvious that $B_1 = 3$, $B_2 = 3$.

For three colors, consider a tetrahedron. There are three sets of two skew edges. If we color each pair by a different color, and both edges in a pair by that same color, we get a complete graph of four vertices in three colors, without any almost monochromatic triangle. So, $B_3 > 4$. Now, in a set of three colors, all are neighbors of each other. So, if a triangle has two sides of the same color, it must be almost monochromatic. Now, from a point, at most three sides can originate which are of different colors. If a fourth side exists, two have the same color. So the number of points cannot exceed 4. The fourth edge (fifth point) guarantees an almost monochromatic triangle. So, $B_3 = 5$. ☐

To prove some lower bounds on different B_n, a method based on the one used in [2] can been used.

Let $U_p = \mathbb{Z}_p - 0$ be the multiplicative subgroup of units in $\mathbb{Z}_p$.

Lemma 7. *The set of residues, H, of n^{th} powers in $\mathbb{Z}_p$ is a subgroup of U_p.*

Proof. Consider $a, b \in U_p$. $a^n, b^n \in H$. Also, $a^n \cdot b^n = (ab)^n \in H$. $a^{-1} = \frac{1}{a} \in U_p$. $(a^n)^{-1} = \frac{1}{a^n} = (\frac{1}{a})^n \in H$. Thus H is a subgroup of U_p. □

Theorem 8. *Let p be a prime and let $H \subset U_p$ be the set of n^{th} power residues of $\mathbb{Z}_p$ such that $-1 \in H$. Suppose that $n|(p-1)$ and there exists generator g of U_p such that the following condition never holds:*

$$\exists\ A \in H \text{ such that } A - 1 \in H, gH \text{ or } g^{n-1}H,$$

then $B_n \geq p + 1$.

Before proving Theorem 8, it is necessary to prove the following lemmas.

Lemma 9. $\{H,\ gH, \ldots,\ g^{n-1}H\}$ *are cosets of H.*

Proof. Suppose

$$g^k H = g^l H$$
$$\Rightarrow\ g^{l-k}H = H$$
$$\Rightarrow\ n|l-k.$$

g^{l-k} is a residue. Clearly all possible residues are

$$\{g^n,\ g^{2n}, \ldots,\ g^{(\frac{p-1}{n})n}\}.$$

The maximum difference is $n-1$, so that if $n|(l-k)$, $l = k$. So all the sets are distinct. As there can only be n cosets, these represent all of them. □

Definition 10. *Residue Classes*
H and its cosets gH, g^2H,..., $g^{n-1}H$ are defined as residue classes.

We now return to the proof of Theorem 8.

Proof. Suppose $\exists$ p, g as in the statement of the proof. Construct a complete graph of p vertices each labeled 0 through $p-1$. For any given edge connecting two vertices i and j, color it with the color corresponding to residue class of $i - j$, i.e., if $i - j \in g^l H$, it is assigned color k_{l+1}. As all the $g^l H$ are cosets (by Lemma 9), the color for each edge is uniquely defined. As $-1 \in H$, $i - j$ and $j - i$ belong to the same residue class. Suppose that the graph generated above using residue classes contains an almost monochromatic triangle DEF (D, E and F denote both the points as well as the vertex numbers), with edges DE and DF of the same color. Replace every vertex i in the graph by $i - D$. Clearly the colors remain same. Now multiply all vertices by $\frac{1}{F-D}$. Let $\frac{1}{F-D} \in g^k H$. Then for $x \in g^l H$, $\frac{x}{F-D} \in g^{k+l}H$. In other words, **colors have been cyclically permuted so that the notion of neighboring colors is preserved**. Let $\frac{E-D}{F-D} = A$. So the triangle has vertices 0, 1 and A, with the edges joining 0 to 1 and 0 to A of the same color. So $A \in H$ and $(A-1) \in H$, gH or $g^{n-1}H$. This contradicts the assumptions in the statement of the theorem. This shows that the graph colored using the above algorithm does not have an almost monochromatic triangle. So $B_n \geq p + 1$. □

Remark 11. *It is quite evident that $B_{n-1} \leq B_n$, since if B_n points guarantee an almost monochromatic triangle in $n-1$ colors, they will also guarantee one in n colors. Also, $A_{n-1} \leq A_n$.*

It is non-trivial to calculate the nth power residue sets by hand, especially for large n. However, a small computer program can calculate these residue sets easily. Here, a C++ program was written, which takes into account all the conditions in the above theorems and gives out the lower and upper bounds directly, along with g and the residue sets. As no upper bounds can be calculated for $n > 11$ anyway, this has been currently run for only $n = 1$ to $n = 30$. Further computation is hoped to be undertaken shortly. The results are tabulated in Appendix A.

4 Some Results

We prove the following results in further theorems:

$$\begin{aligned} B_1 &= 3 \\ B_2 &= 3 \\ B_3 &= 5 \\ B_4 &\leq 10 \\ B_5 &= 12 \\ B_6 &\leq 32 \\ 30 \leq B_7 &\leq 65 \\ 30 \leq B_8 &\leq 162 \\ 74 \leq B_9 &\leq 353 \\ B_{10} &\leq 982 \\ 90 \leq B_{11} &\leq 2609. \end{aligned} \tag{2}$$

Putting $n = 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10$ and 11 in (1), we get the upper bounds.

Theorem 12. *$B_5 \geq 12$, $B_7 \geq 30$, $B_9 \geq 74$, $B_{11} \geq 90$, $30 \leq B_8 \leq 162$.*

Proof. Using the method mentioned above, taking $n = 5$, $g = 4$ and $p = 11$, we get the quintic residue classes $\mod 11$ as:

$$\{1,\ 10\}$$
$$\{4,\ 7\}$$
$$\{5,\ 6\}$$
$$\{2,\ 9\}$$
$$\{3,\ 8\}.$$

Visibly, the residue classes follow the condition given viz.,

$$\nexists\ A \in H \text{ such that } (A-1) \in H,\ gH \text{ or } g^{n-1}H.$$

Hence, using the coloring scheme illustrated above, there is no almost monochromatic triangle in a graph of 11 points colored in 5 colors. So $B_5 = 12$.

Similarly, for R_7, consider the heptic residues in the field $\mathbb{Z}_{29}$ and let $g = 3$. The residue classes are as given below.

$$\{1,\ 12,\ 17,\ 28\}$$
$$\{3,\ 7,\ 22,\ 26\}$$
$$\{8,\ 9,\ 20,\ 21\}$$
$$\{2,\ 5,\ 24,\ 27\}$$
$$\{6,\ 14,\ 15,\ 23\}$$
$$\{11,\ 13,\ 16,\ 18\}$$
$$\{4,\ 10,\ 19,\ 25\}.$$

It can be seen that there can exist no almost monochromatic triangle in a complete graph of 29 points in 7 colors. So $B_7 \geq 30$. For $n = 9$, take $p = 73$ and $g = 4$. The residue classes are:

$$\{1,\ 10,\ 22,\ 27,\ 46,\ 51,\ 63,\ 72\}$$
$$\{4,\ 15,\ 33,\ 35,\ 38,\ 40,\ 58,\ 69\}$$
$$\{6,\ 13,\ 14,\ 16,\ 57,\ 59,\ 60,\ 67\}$$
$$\{9,\ 17,\ 21,\ 24,\ 49,\ 52,\ 56,\ 64\}$$
$$\{5,\ 11,\ 23,\ 36,\ 37,\ 50,\ 62,\ 68\}$$
$$\{2,\ 19,\ 20,\ 29,\ 44,\ 53,\ 54,\ 71\}$$
$$\{3,\ 7,\ 8,\ 30,\ 43,\ 65,\ 66,\ 70\}$$
$$\{12,\ 26,\ 28,\ 32,\ 41,\ 45,\ 47,\ 61\}$$
$$\{18,\ 25,\ 31,\ 34,\ 39,\ 42,\ 48,\ 55\}.$$

So $B_9 \geq 74$. For $n = 11$, take $p = 89$ and $g = 9$. The residue classes are:

$$\{1,\ 12,\ 34,\ 37,\ 52,\ 55,\ 77,\ 88\}$$
$$\{9,\ 19,\ 23,\ 39,\ 50,\ 66,\ 70,\ 80\}$$
$$\{5,\ 7,\ 8,\ 29,\ 60,\ 81,\ 82,\ 84\}$$
$$\{6,\ 17,\ 26,\ 44,\ 45,\ 63,\ 72,\ 83\}$$
$$\{25,\ 33,\ 35,\ 40,\ 49,\ 54,\ 56,\ 64\}$$
$$\{4,\ 30,\ 41,\ 42,\ 47,\ 48,\ 59,\ 85\}$$
$$\{3,\ 13,\ 22,\ 36,\ 53,\ 67,\ 76,\ 86\}$$
$$\{20,\ 27,\ 28,\ 32,\ 57,\ 61,\ 62,\ 69\}$$
$$\{2,\ 15,\ 21,\ 24,\ 65,\ 68,\ 74,\ 87\}$$
$$\{11,\ 18,\ 38,\ 43,\ 46,\ 51,\ 71,\ 78\}$$
$$\{10,\ 14,\ 16,\ 31,\ 58,\ 73,\ 75,\ 79\}.$$

So $B_{11} \geq 90$.

If we try to follow the method for $n = 8$, we get $p = 17$ and $g = 3$, the residue classes being:

$$\{1,\ 16\}$$
$$\{3,\ 14\}$$
$$\{8,\ 9\}$$
$$\{7,\ 10\}$$
$$\{4,\ 13\}$$
$$\{5,\ 12\}$$
$$\{2,\ 15\}$$
$$\{6,\ 11\}.$$

Thus we see that the lower bound computed for $n = 8$ is less than that computed for $n = 7$. As this is scarcely possible, it must be acknowledged that the lower bound for $n = 8$ is worthless and must be replaced by that for $n = 7$. So

$$30 \leq B_8 \leq 162. \tag{3}$$

Unfortunately, the method fails to generate lower bounds in case of $n = 4,\ 6$ and 10. □

Combining results, we get (2). Other bounds, as calculated on a computer, are given in Appendix A.

5 Variation of the Problem

Another problem related to the Ramsey Numbers is studied by Vania Mascioni [3]. This considers a finite metric space with distances between the points belonging to the set $\{1,\ 2,\ \ldots,\ n\}$. The problem is to determine the number of points such that the metric space does not contain an equilateral triangle. This number is denoted by D_n. The known values of D_n are:

$$\begin{aligned} D_1 &= 3 \\ D_2 &= 6 \\ D_3 &= 12 \\ D_4 &= 33 \\ 81 \leq &D_5 \leq 95 \\ 251 \leq &D_6 \leq 389 \\ 551 \leq &D_7 \leq 1659. \end{aligned} \tag{4}$$

Now, a generalization of *this* problem can be concocted, which has a similar relationship to Mascioni's problem as the almost-monochromatic triangle problem has to Ramsey's Problem. An *almost equilateral triangle* can be defined as follows:

Definition 13. *Almost Equilateral Triangle*
A triangle is defined to be almost equilateral if its side-lengths differ by at most one unit.

Notice that an almost equilateral triangle is necessarily isosceles. Now the following problem presents itself: How many points should a metric space contain such that it does not contain an almost equilateral triangle?

Definition 14. C_n
We define C_n to be the smallest integer m such that any finite metric space (= fms) consisting of m points and with distances in a given set must contain an almost equilateral triangle.

The almost equilateral problem is very closely related to the almost monochromatic problem.

To study C_n for different values of n, we need to introduce the following concepts.

Definition 15. *k-Neighborhood*
The k-Neighborhood of a point is defined as the set of all points at a distance k from it.

This differs from the almost monochromatic color spectrum problem in that some triangles are not allowed because the triangle inequality has to be followed. Thus

Remark. $C_n \leq A_n$ for all values of n, so that the upper bounds for A_n are therefore the upper bounds for C_n.

Some elementary results regarding almost equilateral triangles are easy to derive and are given below.

Theorem 16. $C_1 = 3,\ C_2 = 3,\ C_3 = 5.$

Proof. If the distances belong to $\{1\}$, then 3 points will always form an equilateral triangle. So $C_1 = 3$. Similarly, if the distances belong to $\{1,\ 2\}$, then 3 points will always form an almost equilateral triangle. So $C_2 = 3$. If the distances belong to $\{1,\ 2,\ 3\}$, consider the 1-neighborhood of point P. There can only be one point in this neighborhood, for, if there are two or more, they can only be at distance 3 from each other to prevent an almost equilateral triangle, which violates the triangle inequality. Similarly, there can be only one point in the 2-neighborhood of P. The 3-neighborhood of P can only be a metric space with distances belonging to $\{1\}$, which has already derived to consist of maximum 2 points. Consider these as A and B and the point in 2-neighborhood of P as C. Now A and B can only be at distance 1 from each other, to avoid an almost equilateral triangle. So is the case with A and C, as well as B and C. Thus we are forced to conclude that A, B and C form an equilateral triangle. Thus, $C_3 \leq 5$. Now consider a tetrahedron, which has 6 pairs of skew edges. Let each pair be of the same length. This creates a complete graph of 4 points in $\{1,\ 2,\ 3\}$ metric space without an almost equilateral triangle. Hence, $C_3 = 5$. □

Theorem 17. $C_4 \leq 8.$

Proof. As in Theorem 16, the 1-neighborhood of P in case the distances belong to $\{1,\ 2,\ 3,\ 4\}$ can contain 1 point, the 2-neighborhood can contain 2 points at a distance 4 from each other. The 3-neighborhood is again a $\{1\}$ and the 4-neighborhood is a $\{1,\ 2\}$ space, both of which can contain 2 points. Therefore the maximum number of points without almost equilateral triangles is $1 + 1 + 2 + 2 + 1 = 7$. So $C_4 \leq 8$. □

6 Conclusion

The study of the original Ramsey Numbers R_n and its variations A_n, B_n and C_n shows that all the different concepts are very closely related. In several cases, expressions for one contain the others. We conjecture that all these three numbers for different values of n may end up supplementing each other, in that knowing the values for some of them, all other values can be calculated.

7 Further Work

The further research that is proposed includes studying more of the properties of the original Ramsey Numbers R_n and its variations A_n, B_n and C_n. As the upper bound for B_n depends on the upper bound for the original Ramsey Numbers, the upper bounds for many of the higher values of n could not be found out. Some further work on either extending the knowledge about Ramsey Numbers or trying to make the formula for upper bound of B_n independent of the Ramsey Numbers is proposed.

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References

[1] F. P. Ramsey: On a problem of formal logic, *Proc. London Math. Soc.* (2) 30 (1930) 264 – 286.

[2] R. E. Greenwood, A. M. Gleason: Combinatorial relations and chromatic graphs, *Canad. J. Math.* 7 (1955) 1 7.

[3] Vania Mascioni: Equilateral triangles in finite metric spaces, *The Electronic Journal Of Combinatorics* 11 (2004), #R18.

A Appendix: Calculating the Bounds on a Computer

Upper and lower bounds for B_n calculated on a computer for $n = 1$ to $n = 30$ (if a value is not given, it could not be found by the method described):

$$\begin{aligned}
B_1 &= 3\\
B_2 &= 3\\
B_3 &= 5\\
B_4 &\leq 10\\
B_5 &= 12\\
B_6 &\leq 32\\
30 \leq B_7 &\leq 65\\
30 \leq B_8 &\leq 162\\
74 \leq B_9 &\leq 353\\
B_{10} &\leq 982\\
90 \leq B_{11} &\leq 2609\\
132 \leq B_{13}&\\
282 \leq B_{14}&\\
572 \leq B_{15}&\\
930 \leq B_{16}&\\
444 \leq B_{17}&\\
578 \leq B_{18}&\\
762 \leq B_{19}&\\
674 \leq B_{21}&\\
1290 \leq B_{23}&\\
1250 \leq B_{24}&\\
1602 \leq B_{25}&\\
1614 \leq B_{26}&\\
1974 \leq B_{29}&.
\end{aligned}$$